The Problem Corner

Edited by Pat Costello and Kenneth M. Wilke

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solution should not demand any tools beyond calculus and linear algebra. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following new problems should be submitted on separate sheets before July 1, 2007. Solutions received after this will be considered up to the time when copy is prepared for publication. The solutions received will be published in the Fall, 2007 issue of The Pentagon. Preference will be given to correct student solutions. Affirmation of student status and school should be included with solutions. New problems and solutions to problems in this issue should be sent to Pat Costello, Department of Mathematics and Statistics, Eastern Kentucky University, 521 Lancaster Avenue, Richmond, KY 40475-3102 (e-mail: pat.costello@eku.edu, fax: (859)-622-3051)

NEW PROBLEMS 604-610

Problem 604. Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI.

Find the sum $\sum_{n=1}^{\infty} (-1)^{n-1} \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n - \gamma \right)$, where $\gamma$ is the Euler-Mascheroni constant.

Problem 605. Proposed by Cathy George (student) and Russell Euler, Northwest Missouri State University, Maryville, MO.

Geometrically, centered triangular numbers consist of a central dot with three dots around it and then additional dots in the gaps between adjacent dots (see figure). The first four centered triangular numbers are 1, 4, 10, 19. Prove that a positive integer $m \geq 5$ is a centered triangular number if and only if $m$ is the sum of three consecutive triangular numbers.
Problem 606. Proposed by Mathew Cropper and Bangteng Xu (jointly), Eastern Kentucky University, Richmond, KY.

Show that for any nonnegative integer \( n \),

\[
a_n = \frac{2}{7} (-1)^n + \frac{1}{2 (4 + \sqrt{2})} \left( \frac{2}{2 + \sqrt{2}} \right)^{n+1} + \frac{1 + 2\sqrt{2}}{14\sqrt{2}} \left( \frac{2}{2 - \sqrt{2}} \right)^{n+1}
\]

is an integer, and \( a_n \) is even when \( n \geq 2 \).

Problem 607. Proposed by Russell Euler and Jawad Sadek (jointly), Northwest Missouri State University, Maryville, MO.

A point \( P \) is moving on a quarter circle with center \( O \), and \( P \) is bounded by two points \( A \) and \( B \). Let \( PQ \) be the perpendicular from \( P \) to the radius \( OA \). Choose a point \( M \) on the ray \( OP \) such that

\[
\text{length of } OM = \text{length of } OQ + \text{length of } QP.
\]

Find the locus of the points \( M \) as \( P \) moves on the quarter circle.
Problem 608. Proposed by Russell Euler and Jawad Sadek (jointly), Northwest Missouri State University, Maryville, MO.

Let $C$ be a circle with center $O$ and radius $R$. Let $ABCD$ be a parallelogram circumscribed about $C$. Express

$$\frac{1}{(AC)^2} + \frac{1}{(BD)^2}$$

in terms of $R$.

Problem 609. Proposed by Mathew Cropper, Eastern Kentucky University, Richmond, KY.

Let the $n$ vertices of a given graph $G$ be labeled $v_1, v_2, ..., v_n$. Form a new graph $M(G)$ from $G$ in the following way:

1. Add to $G$ an additional $n + 1$ vertices, $u_1, u_2, ..., u_n, w$.
2. Connect each vertex $u_i$ by a new edge to every vertex $v_j$ where there is an edge from $v_i$ to $v_j$ in $G$.
3. Connect each vertex $u_i$ by an edge to vertex $w$. Join each vertex $u_i$ by an edge to vertex $w$.

Starting with a graph which is just two vertices and one edge between the two vertices, find a formula for the number of edges in the iterated graph $M^{(n)}(G) = M(M(M(\cdots(M(G))))$, where the graph formation is iterated $n$ times.

Problem 610. Proposed by the editor.

Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + 4$, where all coefficients $a_i$ are positive reals. If $f(x)$ has $n$ real roots, prove that $f(1) \geq 2^{n+1}$.

Please help your editor by submitting problem proposals.
**Problem 585.** (Corrected) Proposed by José Luis Diaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.

Suppose that the roots $z_1, z_2, \ldots, z_n$ of

$$
z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1z + a_0 = 0
$$

are in arithmetic progression with difference $d$. Prove that

$$
d^2 = 12 \left[ \frac{(n-1)a_{n-1}^2 - 2na_{n-2}}{n^2(n^2 - 1)} \right].
$$

**Solution by the proposer.**

From the Cardan-Viète formulae, we have

$$
z_1 + z_2 + \ldots + z_n = -a_{n-1}
$$

and

$$
z_1^2 + z_2^2 + \ldots + z_n^2 = a_{n-1}^2 - 2a_{n-2}.
$$

The expression for $d^2$ can be rewritten as

$$
a_{n-1}^2 - 2a_{n-2} = \left( \frac{1}{n} \right) a_{n-1}^2 + \left[ \frac{n(n^2 - 1)}{12} \right] d^2,
$$

or equivalently we need to prove that

$$
z_1^2 + z_2^2 + \ldots + z_n^2 = \frac{(z_1 + z_2 + \ldots + z_n)^2}{n} + \left[ \frac{n(n^2 - 1)}{12} \right] d^2.
$$

In fact,

$$
z_1^2 + z_2^2 + \ldots + z_n^2 = \sum_{k=0}^{n-1} (z_1 + kd)^2
$$

$$
= \sum_{k=0}^{n-1} (z_1^2 + 2z_1kd + k^2d^2)
$$

$$
= nz_1^2 + 2dz_1 \sum_{k=0}^{n-1} k + d^2 \sum_{k=0}^{n-1} k^2
$$

$$
= nz_1^2 + 2dz_1 \cdot \frac{n(n-1)}{2} + d^2 \cdot \frac{n(n-1)(2n-1)}{6}
$$

$$
= nz_1^2 + 2dz_1 \cdot \frac{n(n-1)}{2} + d^2 \cdot \frac{n(n-1)(2n-1)}{6}
$$
\[
\left( z_1 + \frac{(n-1)d}{2} \right)^2 + \frac{n(n^2-1)d^2}{12} = \frac{(z_1 + z_2 + \ldots + z_n)^2}{n} + \frac{n(n^2-1)d^2}{12},
\]
and we are done.

Editor’s Comment: The Cardan-Viete formulae are also known as elementary symmetric functions of the roots of an equation and Newton’s Relations.

**Problem 587.** (Corrected) Proposed by José Luis Diaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.

Show that if \( A, B, C \) are the angles of a triangle, and \( a, b, c \) its sides, then
\[
\prod_{\text{cyclic}} \sin^{1/3}(A - B) \leq \sum_{\text{cyclic}} \frac{(a^3 + b^3) \sin(A - B)}{3ab}.
\]

**Solution by the proposer.**

From the determinant
\[
\begin{vmatrix}
\cos A & \sin A & \cos A \\
\cos B & \sin B & \cos B \\
\cos C & \sin C & \cos C
\end{vmatrix} = 0,
\]
expanding and re-ordering the terms, we obtain the identity
\[
\cos A \sin (B - C) + \cos A \sin (B - C) + \cos A \sin (B - C) = 0. \quad (1)
\]
Using the Law of Cosines, we obtain \( \cos A = \frac{b^2 + c^2 - a^2}{2bc} \). Similarly, \( \cos B = \frac{c^2 + a^2 - b^2}{2ac} \), and \( \cos C = \frac{a^2 + b^2 - c^2}{2ab} \). Putting these results into (1) yields
\[
\left( \frac{b^2 + c^2 - a^2}{2bc} \right) \sin (B - C) + \left( \frac{c^2 + a^2 - b^2}{2ac} \right) \sin (C - A) + \left( \frac{a^2 + b^2 - c^2}{2ab} \right) \sin (A - B) = 0,
\]
or equivalently,
\[ a(b^2 + c^2) \sin(B - C) + b(c^2 + a^2) \sin(C - A) + c(a^2 + b^2) \sin(A - B) = a^3 \sin(B - C) + b^3 \sin(C - A) + c^3 \sin(A - B). \]

Applying the AM-GM Inequality to the RHS of (3) yields

\[ a^3 \sin(B - C) + b^3 \sin(C - A) + c^3 \sin(A - B) \geq 3abc \left( \prod_{\text{cyclic}} \sin(A - B) \right)^{1/3}. \]

Therefore

\[ a(b^2 + c^2) \sin(B - C) + b(c^2 + a^2) \sin(C - A) + c(a^2 + b^2) \sin(A - B) \geq 3abc \left( \prod_{\text{cyclic}} \sin(A - B) \right)^{1/3}. \]

The desired inequality immediately follows by dividing both sides of the preceding inequality by \(3abc\). Note that equality holds when \(\Delta ABC\) is equilateral.

**Problem 589. Proposed by the editor.**

Find \(a, b, c, d,\) and \(e\) so that the number

\[ a8b2cd7e3 \]

is divisible by both 73 and 137, where \(a, b, c, d,\) and \(e\) are distinct integers chosen from the set \(\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}\), and \(a > 0\).
Solution by the editor:

Let \( N \) denote the number \( a8b2cd7e3 \). Since \( N \) is divisible by both 73 and 137, \( N \) is also divisible by \( 10001 = 73 \cdot 137 \). Then since any power of 10 is relatively prime to 10001, we will construct a set of smaller numbers which are also divisible by 10001 whenever 10001 divides \( N \).

We construct

\[
N_1 = \frac{N - 3 \cdot 10001}{10} = a8b2\{c - 3\}d7e
\]

which is clearly divisible by 10001 whenever 10001 divides \( N \). Proceeding similarly we construct

\[
N_2 = \frac{N_1 - e \cdot 10001}{10} = a8b\{2 - e\}\{c - 3\}d7;
\]

\[
N_3 = \frac{N_2 - 7 \cdot 10001}{10} = a8\{b - 7\}\{2 - e\}\{c - 3\}d;
\]

\[
N_4 = \frac{N_3 - d \cdot 10001}{10} = a\{8 - d\}\{b - 7\}\{2 - e\}\{c - 3\};
\]

and finally

\[
N_5 = \frac{N_4 - (c - 3) \cdot 10001}{10} = \{a + 3 - c\}\{8 - d\}\{b - 7\}\{2 - e\}.
\]

Now since \( N_5 \) is divisible by 10001, \( N_5 = 0 \) and either each digit must be zero or \( N_5 = 1001 \) so that \( \{2 - e\} = 1 \) and \( \{a + 3 - c\} = 10 \) simultaneously.

* Suppose that \( N_5 = 0 \). Then \( b = 7, d = 8, e = 2 \) and \( c = a + 3 \), and there are six possible values for \( N \) which are 187248723, 287258723, 387268723, 487278723, 587288723, and 687298723, of which only 187248723, 387268723, and 687298723 satisfy the distinctness of \( a, b, c, d, \) and \( e \).

* Otherwise \( N_5 = 1001 \) so that \( \{2 - e\} = 1 \) and \( \{a + 3 - c\} = 10 \) simultaneously. Here \( b = 7, d = 8, e = 1 \) and \( a - 7 = c \) so that \( (a, c) = (9, 2) \) or \( (8, 1) \) yielding two more possible values of \( N \) which are 987228713 and 887218713, of which only 987218713 satisfies the distinctness of \( a, b, c, d \) and \( e \).

Hence the problem has the four solutions:

187248723, 387268723, 687298723, and 987228713.

Also solved by Chad Birch, Eastern Kentucky University, Richmond, KY. A partial solution was received from David Ritter, student, Messiah College, Grantham, PA.
Problem 590. Proposed by José Luis Diaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.

Evaluate the following sum: \[ \sum_{n=1}^{\infty} \left( 4^n \cos^2 \frac{\pi}{2n+2} \right)^{-1} \]

Solution by the proposer.

From the identity
\[
\frac{1}{\sin^2 x} = \frac{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}}{2} = \frac{1}{4 \sin^2 \frac{x}{2} \cos^2 \frac{x}{2}} + \frac{1}{4 \cos^2 \frac{x}{2}},
\]
we get
\[
\frac{1}{4 \cos^2 \frac{x}{2}} = \frac{1}{\sin^2 x} - \frac{1}{4 \sin^2 \frac{x}{2}}.
\]
Applying this recursively, we have
\[
\sum_{n=1}^{N} \frac{1}{4^n \cos^2 \frac{x}{2n}} = \frac{1}{\sin^2 x} - \frac{1}{4 \sin^2 \frac{x}{2}} + \frac{1}{4 \sin^2 \frac{x}{2}} - \frac{1}{4^2 \sin^2 \frac{x}{2}} + \cdots + \frac{1}{4^{N-1} \sin^2 \frac{x}{2N-1}} - \frac{1}{4^N \sin^2 \frac{x}{2N}},
\]
and
\[
\sum_{n=1}^{\infty} \frac{1}{4^n \cos^2 \frac{x}{2n}} = \lim_{N \to \infty} \left( \frac{1}{\sin^2 x} - \frac{1}{4^N \sin^2 \frac{x}{2N}} \right).
\]
Setting \( x = \frac{\pi}{4} \) in this expression and using L’Hopital’s Rule twice yields
\[
\sum_{n=1}^{\infty} \frac{1}{4 \cos^2 \frac{\pi}{2n+2}} = 2 - \frac{16}{\pi^2} = 2 \left( \frac{\pi^2}{16} - 8 \right).
\]

Also solved by Russell Euler and Jawad Sadek (jointly), Northwest Missouri State University, Maryville, MO.


Express \( \cos A \cos B \sin (A - B) + \cos B \cos C \sin (B - C) + \cos C \cos A \sin (C - A) \) as the product of three sines.
Solution by the proposer.

We have

\[
\begin{align*}
\cos A \cos B \sin (A - B) + \cos B \cos C \sin (B - C) \\
+ \cos C \cos A \sin (C - A) &= \sin (A - B) \sin (B - C) \sin (C - A).
\end{align*}
\]

This problem was inspired by problem 2353 from Crux Mathematicorum [1999, p. 316], which showed as part of the problem that

\[
\begin{align*}
\sin A \sin B \sin (A - B) + \sin B \sin C \sin (B - C) \\
+ \sin C \sin A \sin (C - A) &= \sin (A - B) \sin (B - C) \sin (C - A).
\end{align*}
\]

Just replace each angle in the Crux result by its complement to get the result for the given problem.

Solution by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, MO.

Using the product and factoring formulas of sines and cosines, we have

\[
\begin{align*}
\cos A \cos B \sin (A - B) + \cos B \cos C \sin (B - C) \\
+ \cos C \cos A \sin (C - A) &= \cos A \cos B \sin (A - B) \\
+ \cos C [\cos B \sin (B - C) + \cos A \sin (C - A)] \\
= \cos A \cos B \sin (A - B) \\
+ \frac{1}{2} \cos C [\sin (2B - C) - \sin C + \sin C - \sin (2A - C)] \\
= \cos A \cos B \sin (A - B) \\
+ \frac{1}{2} \cos C [2 \cos (A + B - C) \sin (B - A)] \\
= \cos A \cos B \sin (A - B) + \cos C \cos (A + B - C) \sin (B - A) \\
= \sin (A - B) [\cos A \cos B - \cos C \cos (A + B - C)] \\
= \sin (A - B) [\frac{1}{2} \cos (A + B) + \frac{1}{2} \cos (A - B) \\
- \frac{1}{2} \cos (A + B) - \frac{1}{2} \cos (2C - A - B)] \\
= \frac{1}{2} \sin (A - B) [\cos (A - B) - \cos (2C - A - B)] \\
= \frac{1}{2} \sin (A - B) [-2 \sin (C - B) \sin (A - C)] \\
= - \sin (A - B) \sin (C - B) \sin (A - C) \\
= \sin (A - B) \sin (B - C) \sin (A - C).
\end{align*}
\]

The points \((0, 0), (1, 0), (1, 1),\) and \((0, 1)\) are the vertices of a square \(S\). Find an equation in \(x\) and \(y\) whose graph in the \(xy\)-plane is \(S\).

Solution by the proposer.

Start with the equation \(|x| + |y| = 1\), which is the graph of a diamond. Rotate the axes \(45^\circ\) and perform a scaling to get the equation \(|x - y| + |x + y - 1| = 1\).

Solution by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, MO.

One equation that works is \(\max(x, y) = \min(x + y, 1)\).

Problem 593. Proposed by Kenneth M. Wilke, Washburn University, Topeka, KS.

Let \(N = p^2 + q^2 + r^2 + s^2\), where \(p, q, r,\) and \(s\) are positive integers such that \(pq = rs\). Prove or disprove that \(N\) is prime.

Solution by Carl Libis, University of Rhode Island, Kingston, RI.

The proof is by contradiction. Assume that \(N\) is prime. Since \(p, q, r,\) and \(s\) are positive and \(N\) is prime, we know that \(N\) is an odd prime and that at least one of \(p, q, r,\) and \(s\) is even. Since \(pq = rs\), more than one of \(p, q, r,\) and \(s\) must be even. If exactly two are even, then \(N\) is even and therefore not prime. So exactly three of \(p, q, r,\) and \(s\) are even. Say \(p, r,\) and \(s\) are even, with \(r = 2y,\) and \(s = 2z\). This implies that \(p = 4x\). Thus,

\[
N = 16x^2 + 4y^2 + 4z^2 + q^2 = 16x^2 + 4y^2 + 4z^2 + \frac{y^2z^2}{x^2} = \frac{16x^4 + 4x^2y^2 + 4x^2z^2 + y^2z^2}{x^2} = \frac{(4x^2 + y^2)(4x^2 + z^2)}{x^2}.
\]

Since \(x^2 < 4x^2 + y^2\) and \(x^2 < 4x^2 + z^2\), \(N\) has two non-trivial factors, which contradicts our assumption that \(N\) was prime. Therefore, \(N\) is not prime.
Problem 594. Proposed by the editor.

The sequence \( x_0, x_1, x_2, \ldots \) is defined by the conditions \( x_0 = 0, x_1 = 1, \) and \( x_{n+1} = \frac{n x_n + x_{n-1}}{n+1} \) for \( n \geq 1. \) Determine \( \lim_{n \to \infty} x_n. \)

Solution by Russell Euler and Jawad Sadek (jointly), Northwest Missouri State University, Maryville, MO.

Since the coefficients of \( x_n \) and \( x_{n-1} \) are positive and add up to 1, one can show that the given sequence is bounded, the subsequence with even indices is increasing, and the subsequence with odd indices is decreasing, and the two subsequences converge to the same limit, \( L. \) Now write

\[
x_{n+1} = \frac{n}{n+1} x_n + \frac{1}{n+1} x_{n-1}
\]

\[
= \frac{n}{n+1} \left( \frac{n-1}{n} x_{n-1} + \frac{1}{n} x_{n-2} \right) + \frac{1}{n+1} x_{n-1}
\]

\[
= \frac{n}{n+1} x_{n-1} + \frac{1}{n+1} x_{n-2}.
\]

The second and last equalities imply that

\[
x_n - x_{n-1} = \frac{x_{n-2} - x_{n-1}}{n}.
\]

Taking into account that \( x_0 < x_2 < x_4 < \cdots < x_{2n} < \cdots < L \) and \( x_1 > x_3 > x_5 > \cdots > x_{2n+1} > \cdots > L, \) an induction argument gives

\[
x_n - x_{n-1} = \frac{(-1)^n}{(n+1)!} \quad (*)
\]

Setting \( S_{2n} = (x_2 - x_0) + (x_4 - x_2) + \cdots + (x_{2n} - x_{2n-2}) = x_{2n}, \) we get \( \lim_{n \to \infty} S_{2n} = \lim_{n \to \infty} x_{2n} = \lim_{n \to \infty} x_n = L. \) Finally,

\[
L = \lim_{n \to \infty} S_{2n}
\]

\[
= \lim_{n \to \infty} \left[ (x_2 - x_1) + (x_1 - x_0) + (x_4 - x_3) + (x_3 - x_2) + \cdots + (x_{2n} - x_{2n-1}) + (x_{2n-1} - x_{2n-2}) \right]
\]

\[
= \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{(2n)!} + \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{(2n-1)!} \text{ by (*)}
\]

\[
= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}
\]

\[
= 1 - e^{-1}.
\]

Also solved by the proposer.
**Problem 595.** Proposed by the editor.

Place the first 10 odd primes $3, 5, 7, 11, 13, 17, 19, 23, 29, 31$ into the circles on the pentagon so that the sum of the entries on each side is the same.

[Thanks to Robert Buskirk for assistance with the diagram.]

**Solution** by Sr. Marcella Louise Wallowicz CSFN, Holy Family University, Philadelphia, PA.

Also solved by Carl Libis, University of Rhode Island, Kingston, RI and Sam Bailie, Eastern Kentucky University, Richmond, KY.
Problem 596. Proposed by the editor.

A phone number has the lazy-finger property when the next number that you dial (on a touch-tone phone) is either the same number or a number up or down one button or a number to the left or right one button. For example, 555-2365 has the lazy-finger property. How many 7-digit phone numbers that start with a 5 have the lazy-finger property?

Solution by the proposer.

Let \( p(m, n) \) be the number of \((n + 1)\)-digit phone numbers that have the lazy-finger property and start with \( m \). In particular, \( p(5, 1) \) counts the phone numbers 52, 54, 55, 56, 58. Hence \( p(5, 1) = 5 \). Using this function and a computer program that counts the number of possibilities that can be reached from each particular button, one obtains \( p(5, 6) = 5033 \). There are 5033 phone numbers that start with 5 and have the lazy-finger property.